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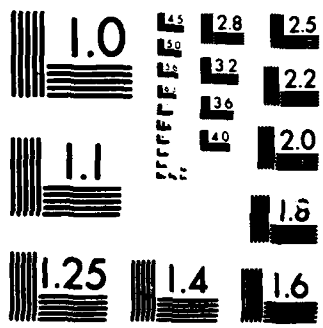

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A PROOF OF GRACE'S THEOREM BY INDUCTION

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ABSTRACT

Two polynomials in $\mathbb{C}[z]$

$$(1) \qquad A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k , \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are said to be apolar, provided that the equation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0$$

Grace's theorem states:

holds. This definition was given at the ~~turn~~ of the century by J.H. Grace who established in [1] the ~~following~~

Theorem of Grace. Let the polynomials (1) be apolar. If the circular region C contains all the zeros of $A(z)$, then C must contain at least one of the zeros of $B(z)$.

By a circular region ~~we mean~~ is meant either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

~~Here we~~ give a proof of Grace's theorem by mathematical induction on the degree n.

AMS (MOS) Subject Classifications: 30C10, 30C15

Key Words: Zeros of polynomials; Möbius transformations.

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

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SIGNIFICANCE AND EXPLANATION

Two polynomials in $\mathbb{C}[z]$

$$(1) \qquad A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k, \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are said to be *apolar*, provided that the equation

$$(2) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0$$

holds. This definition was given at the turn of the century by J.H. Grace who established in [1] the following

Theorem of Grace. *Let the polynomials* (1) *be apolar. If the circular region* C *contains all the zeros of* A(z), *then* C *must contain at least one of the zeros of* B(z).

By a circular region we mean either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

Here we give a proof of Grace's theorem by mathematical induction on the degree n.

The references [3] and [2] give numerous applications of Grace's theorem. For $n = 2$ the apolarity equation (1.2) is equivalent to the equation

$$\frac{\beta_2-\alpha_2}{\beta_2-\alpha_1} : \frac{\beta_1-\alpha_2}{\beta_1-\alpha_1} = -1,$$

hence the pair of points (β_1,β_2) divides (α_1,α_2) in harmonic ratio.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

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A PROOF OF GRACE'S THEOREM BY INDUCTION

A.W. Goodman and I.J. Schoenberg

1. Introduction. At the turn of the century J.H. Grace [1] introduced the following

Definition 1. Two polynomials

$$A(z) = a_0 + \binom{n}{1}a_1 z + \cdots + \binom{n}{k}a_k z^k + \cdots + a_n z^n \tag{1.1}$$

and

$$B(1) = b_0 + \binom{n}{1}b_1 z + \cdots + \binom{n}{k}b_k z^k + \cdots + b_n z^n \tag{1.2}$$

are said to be apolar provided that their coefficients satisfy the apolarity condition

$$a_0 b_n - \binom{n}{1}a_1 b_{n-1} + \cdots + (-1)^k\binom{n}{k}a_k b_{n-k} + \cdots + (-1)^n a_n b_0 = 0. \tag{1.3}$$

The coefficients of the polynomials may be real or complex. If $a_r \neq 0$ $(r \geq 0)$ and $a_\nu = 0$ for $\nu = r+1, r+2, \ldots, n$, then we regard $z = \infty$ as an $(n-r)$-fold zero of $A(z)$. If all the coefficients of $A(z)$ are zero, then $A(z)$ is not regarded as a polynomial.

Grace discovered the following remarkable

Theorem of Grace. Let the polynomials (1.1) and (1.2) be apolar. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of $A(z)$ and $\beta_1, \beta_2, \ldots, \beta_n$ be the zeros of $B(z)$. If the circular region C contains all of the α_ν, then C must contain at least one of the β_ν.

By a circular region we mean either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

In [3] G. Szegö gave a proof of Grace's theorem freed of the invariant-theoretic concepts used by Grace in [1], and he also gave a large number of applications. In the present note we establish Grace's theorem by induction on n. Our proof is different from those given earlier.

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2. The invariance of apolarity by Möbius transformations.

By the transform of $A(z)$ under the Möbius transformation

(2.1) $$z = \frac{aw+b}{cw+d} \quad (ad-bc \neq 0)$$

we mean the polynomial

$$A^*(w) \equiv (cw+d)^n A\left(\frac{aw+b}{cw+d}\right) \equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu (aw+b)^\nu (cw+d)^{n-\nu} \equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu.$$

For example if $A(z) \equiv 1$, then $A^*(w) = (cw+d)^n$ and the n-fold zero of $A(z)$ at $z = \infty$ becomes an n-fold zero of $A^*(z)$ at $w = -d/c$ if $c \neq 0$.

Lemma 1. Let $A(z)$ and $B(z)$ be apolar polynomials. If the Möbius transormation (2.1) changes the polynomials (1.1) and (1.2) into

(2.2) $$A^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu \quad \text{and} \quad B^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu^* w^\nu,$$

then the polynomials (2.2) are also apolar.

Proof. It sufficies to prove Lemma 1 for each of the three special transformations

(2.3) $$\text{(i)} \quad z = w + h, \quad \text{(ii)} \quad z = kw, \quad \text{(iii)} \quad z = \frac{1}{w} .$$

(i) $$A^*(w) = A(w + h) = \sum_{\nu=0}^{n} \frac{w^\nu}{\nu!} A^{(\nu)}(h)$$

and therefore

$$A^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(n-\nu)!}{n!} A^{(\nu)}(h) w^\nu.$$

Similarly

$$B^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(n-\nu)!}{n!} B^{(\nu)}(h) w^\nu.$$

The apolarity equation for these polynomials is

$$f(h) \equiv \sum_{\nu=0}^{n} (-1)^{\nu}\binom{n}{\nu} \frac{(n-\nu)!}{n!} A^{(\nu)}(h)\frac{\nu!}{n!} B^{(n-\nu)}(h) = 0$$

or

$$n!\, f(h) = \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu)}(h) B^{(n-\nu)}(h) = 0. \tag{2.4}$$

The apolarity of $A(z)$ and $B(z)$ gives $f(0) = 0$, and we must show that $f(h) = 0$ for all h. This will follow as soon as we show that for all h

$$f'(h) = 0. \tag{2.5}$$

From (2.4) we find that

$$n!\, f'(h) = \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu+1)}(h) B^{(n-\nu)}(h) + \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu)}(h) B^{(n-\nu+1)}(h).$$

Here the νth term $(\nu<n)$ in the first sum cancels with the $(\nu+1)$-st term in the second term, and hence

$$n!\, f'(h) = (-1)^n A^{(n+1)}(h)B(h) + A(h)B^{(n+1)}(h)$$

which is evidently zero because $A(z)$ and $B(z)$ are nth degree polynomials. This proves (2.5) and therefore (2.4) for all h.

(ii) For the second transformation in (2.3) we have

$$A^*(w) = a_0 + \binom{n}{1} a_1 kw + \cdots + a_n k^n w^n,$$

and

$$B^*(w) = b_0 + \binom{n}{1} b_1 kw + \cdots + b_n k^n w^n$$

which are evidently apolar by (1.3).

(iii) Finally, setting $z = 1/w$ gives

$$A^*(w) = a_n + \binom{n}{1} a_{n-1} w + \cdots + a_0 w^n$$

and

$$B^*(w) = b_n + \binom{n}{1}b_{n-1}w + \cdots + b_0w^n$$

and these are also apolar by (1.3). ∎

Lemma 2. If α is a zero of the polynomial $A(z)$, then its transform β under (2.1) is a zero of the transformed polynomial $A^*(w)$.

If neither α nor β is ∞, then $\alpha = (a\beta+b)/(c\beta+d)$ and

$$A^*(\beta) = (c\beta+d)^n A\left(\frac{a\beta+b}{c\beta+d}\right) = (c\beta+d)^n A(\alpha) = 0. \tag{2.6}$$

If $\alpha = \infty$ is an r-fold zero of $A(z)$, then $\beta = -d/c$ is clearly an r-fold zero of $A^*(z)$. If $\alpha = a/c$ is an r-fold zero of $A(z)$, then the decomposition used in the proof of Lemma 1 shows that $\beta = \infty$ is an r-fold zero of $A^*(z)$. ∎

It follows from Lemma 2 that if a circular domain C contains all the zeros of $A(z)$ then the transformed domain under (2.1) will contain all the zeros of $A^*(z)$.

3. Proof of Grace's Theorem. We use induction on n. For $n = 1$, the apolarity condition (1.3) gives $a_0b_1 - a_1b_0 = 0$ so $\alpha_1 = \beta_1$ and the theorem is obviously true.

Next we assume the theorem is true for index $n-1$ and wish to prove that it is also true for index n. Here we use the method of contradiction. We shall assume that for some circular domain C and some pair of apolar polynomials $A(z)$ and $B(z)$

$$\alpha_\nu \in C, \quad \nu = 1,2,\ldots,n, \quad \text{and } \beta_\nu \notin C, \quad \nu = 1,2,\ldots,n. \tag{3.1}$$

By a transformation we may assume that $\beta_n = \infty$, without loss of generality (use Lemmas 1 and 2). It follows that in (1.2)

$$b_n = 0. \tag{3.2}$$

The second assumption in (3.1) tells us that $\beta_n \notin C$ and hence C is bounded. Therefore all α_ν are finite and hence $a_n \neq 0$. The points $\beta_1, \beta_2, \ldots, \beta_{n-1}$ (finite or not) are the zeros of

$$B(z) = b_0 + \binom{n}{1}b_1z + \cdots + \binom{n}{k}b_kz^k + \cdots + \binom{n}{n-1}b_{n-1}z^{n-1} \tag{3.3}$$

which we now regard as a polynomial of degree $n-1$. Now consider the polynomial

$$\frac{1}{n}A'(z) = a_1 + \binom{n-1}{1}a_2z + \cdots + \binom{n-1}{k}a_{k+1}z^k + \cdots + a_nz^{n-1} \tag{3.4}$$

having the zeros $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. These zeros are all finite because $a_n \neq 0$.

We claim the two polynomials (3.3) and 3.4) are apolar as polynomials of degree

$n-1$. To confirm this we rewrite (3.3) in the usual form

$$B(z) = b_0' + \binom{n-1}{1} b_1' z + \cdots + \binom{n-1}{k} b_k' z^k + \cdots b_{n-1}' z^{n-1}. \tag{3.5}$$

Then

$$\binom{n}{k} b_k = \binom{n-1}{k} b_k', \quad k = 0,1,2,\ldots,n-1. \tag{3.6}$$

But then our original apolarity condition (1.3)

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} b_k a_{n-k} = 0$$

(since $b_n = 0$ by (3.2)) becomes

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} b_k' a_{n-k} = 0.$$

This shows that the polynomials (3.4) and (3.5) are apolar.

We now appeal to the Gauss-Lucas Theorem which states that all the zeros $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are in the convex hull of the zeros $\alpha_1, \alpha_2, \ldots \alpha_n$ of $A(z)$. By our first assumption (3.1) we conclude that $\gamma_\nu \in C$, for $\nu = 1,2,\ldots,n-1$. On the other hand $\beta_\nu \notin C$ for $\nu = 1,2,\ldots,n-1$. This contradicts Grace's Theorem for index $n-1$. Hence by the principle of mathematical induction Grace's Theorem is true for every positive integer n. ■

The reader is referred to Szegős work [3] and the book by Marden [2] for many interesting applications of Grace's Theorem.

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1. J.H. Grace, The zeros of polynomials, Proc. Cambridge Philos. Soc. 11 (1900-1902) 352-357.

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3. Gabor Szegö, Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen, Math. Zeit. 13 (1922) 28-56.

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